In-flight Estimation of the Cassini Spacecraft's Inertia Tensor

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Summary

A new methodology for estimating the spacecraft 3-by-3 inertia tensor is proposed in this study. This methodology exploits the fact that the total angular momentum vector of a spacecraft, as expressed in an inertial coordinate system, is conserved when it is being slewed about any of its axes by three reaction wheels. Inputs needed for this methodology are telemetry data of the reaction wheel spin rates, spacecraft per-axis angular velocities over the time duration of the slews, and ground-measured inertia properties of the reaction wheels. By collecting these data over one or more per-axis spacecraft slews, one can make least-square estimates of both the moments and products of inertia of the spacecraft. On March 15, 2000, the Cassini spacecraft used its three reaction wheels to perform slews about its three axes. The proposed methodology has been applied on the telemetry data collected on that day. The inertia matrix estimated using the proposed methodology agrees well with that predicted pre-launch. The uncertainties associated with this estimated inertia matrix have also been computed.

Introduction

The Cassini spacecraft was launched on 15 October 1997 by a Titan 4B launch vehicle. After an interplanetary cruise of almost seven years, it will arrive at Saturn in July 2004. To save propellant, Cassini will make several gravity-assist flybys: two at Venus and one each at Earth and Jupiter. Unlike Voyagers 1 and 2, which only flew by Saturn, Cassini will orbit the planet for at least four years. Major science objectives of the Cassini mission include investigations of the configuration and dynamics of Saturn's magnetosphere, the structure and composition of the rings, the characterization of several of Saturn's icy moons, and others. The Huygens probe, developed by the European Space Agency, will be ejected in November 2004 and will study the atmosphere of Titan, the only moon in the solar system with a substantial atmosphere. Fig. 1 depicts the Cassini spacecraft in its "Cruise" configuration.

Cassini's Attitude and Articulation Control Subsystem (AACS) estimates and controls the spacecraft attitude, and responds to ground-commanded pointing goals for the spacecraft's science instrument and/or communication antennas with respect to targets of interest. The AACS also executes ground-commanded spacecraft velocity changes. Hardware that are used by AACS to perform the functions mentioned above include two attitude control flight computers, an accelerometer, four reaction wheel actuators (RWA), two sun sensors, two stellar reference units (star trackers), two inertial reference units (with eight sensing axes), and others. These hardware is controlled using on-board flight software algorithms that reside in the flight computers.

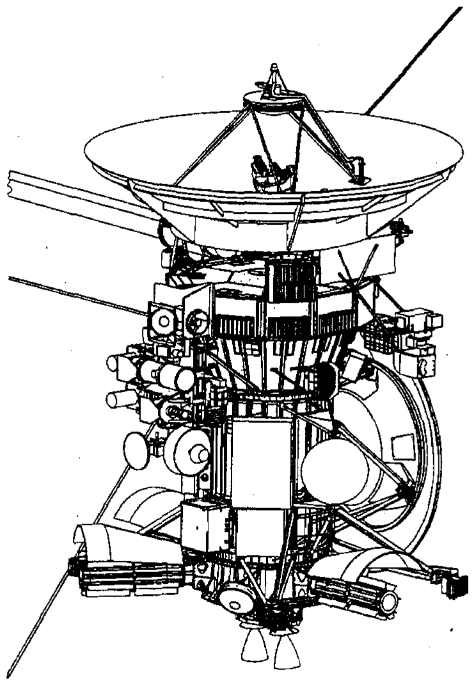

Figure 1 Cassini Cruise Configuration

Several AACS flight software algorithms onboard the spacecraft use knowledge of the Cassini spacecraft 3-by-3 inertia matrix. It is used by both the AACS fault protection algorithms and the Attitude Estimator (in its "gyro-less" attitude determination mode). Thruster vector control algorithms, which are employed to maintain spacecraft attitude control during all propulsive maneuvers performed by the 445-Newton main engine, also make use of the on-board inertia matrix. The inertia tensor is also used by the RWA attitude controller that is used to maintain precision spacecraft attitude control during imaging of science targets. As such, a highly accurate estimate of this inertia matrix is important to spacecraft operations.

Before launch, the inertia matrix of Cassini was estimated by adding together the moments of inertia of the individual components of the spacecraft. The moments of inertia of individual components were computed with respect to the predicted center of mass of the overall spacecraft before being summed. After launch, the onboard spacecraft inertia matrix is updated periodically using estimates of how much propellant (e.g. hydrazine, fuel, and oxidizer) has been used to date. Also, the deployment of the magnetometer boom and the ejection of the Huygens probe change the inertia matrix. These discrete events must therefore be considered in updating the onboard spacecraft inertia matrix. The inertia matrix of the spacecraft on March 15, 2000 (the Magnetometer boom has been deployed but the Huygens Probe had not yet been ejected), using the "sum-of-all-components" method, is estimated to be:

$$\vec{I}_{SC} = \begin{bmatrix} 8810.8 & -136.8 & 115.3 \\ -136.8 & 8157.3 & 156.4 \\ 115.3 & 156.4 & 4721.8 \end{bmatrix} kg \cdot m^2 \tag{1}$$

This method of calculating the inertia tensor had not been validated inflight using an independent approach until this study.

Goals

The primary goal of this paper is to propose and validate a methodology that can be used to estimate the Cassini spacecraft inertia tensor. In addition, the methodology developed could also be used to estimate the inertia tensor of other spacecraft whose attitudes are controlled by reaction wheels.

Problem Formulation

When a spacecraft is slewed using the RWAs, the total angular momentum of the spacecraft expressed in an inertial coordinate frame is conserved. This is because the addition of angular momentum on the spacecraft due to external torque, such as solar radiation torque, is typically very small over the duration of the slew. Over a spacecraft slew, good estimates of the following quantities are available, either from direct measurement prior to launch or from the telemetry data sent down from the spacecraft:

- Spacecraft angular rates (ω_x, ω_y, and ω_z) in a XYZ body coordinate frame
- Spin rates with respect to the spacecraft of the three RWAs (ω_1, ω_2, and ω_3) about the spin axes of their respective assemblies
- Spacecraft quaternion vector (q_1, q_2, q_3, and q_4) which could be used to compute a coordinate transformation matrix from the inertial (J_{2000}) coordinate frame to the XYZ body coordinate frame
- Moments of inertia of the three RWAs (I_{RWA})
- Transformation matrix from the three RWA spin axes to the XYZ body coordinate frame (T).

When the spacecraft is being slewed by the RWAs, there are changes in the both the spacecraft's angular rates and the spin rates of the RWAs, however, the spacecraft still must obey Euler's equation. Since the magnitude of the external torque exerted on the spacecraft about all its axes is negligibly small (see Appendix A), the total angular momentum vector of the spacecraft in an inertial coordinate frame is conserved throughout the maneuver. This total angular momentum vector has two components, one from the spacecraft rates and one from the RWA rates. The conservation of angular momentum allows the total angular momentum evaluated at the initial time (prior to the beginning of the slew) to be set equal to the total angular momentum evaluated throughout the slew. This equality gives us an equation for each time step throughout the slew with only one unknown: I_{SC}, which can then be estimated via a least-squares approach. Note that I_{SC} contains the moments of inertia of the three stationary reaction wheels.

Let H stand for angular momentum. The total angular momentum vector of the spacecraft has two components: $\vec{H}_{Total} = \vec{H}_{Spacecraft} + \vec{H}_{RWAs}$. The component due to the spacecraft rates is: $\vec{H}_{Spacecraft} = [I_{SC}]\vec{\Omega}$, where $\vec{\Omega} = [\omega_x,\ \omega_y,\ \omega_z]^T$. Here, I_{SC} is the unknown inertia tensor of the spacecraft, and ω_x, ω_y, and ω_z are the X, Y, and Z components of the spacecraft angular velocity vector in body coordinate. Since the conservation of angular momentum is only valid in an inertial coordinate system, a transformation matrix, defined here from the J_{2000} inertial frame to the body coordinate frame must be defined. It is computed using the four components of the spacecraft's quaternion vector (q_1, q_2, q_3, and q_4).

$$[P] = \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_2q_1-q_3q_4) & 1-2(q_1^2+q_3^2) & 2(q_2q_3+q_1q_4) \\ 2(q_3q_1+q_2q_4) & 2(q_3q_2-q_1q_4) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

To determine the angular momentum of the RWAs, let us first define $\vec{\rho} = [\omega_1,\ \omega_2,\ \omega_3]^T$. Here, ω_i is the angular rate of RWA_i, i= 1-3. These spin rates are defined with respect to the RWAs' own spin axes. There are four RWAs on board the spacecraft. RWA_1, RWA_2, and RWA_3 make up the primary set. The articulatable RWA_4 can be oriented in the same direction as any of the other three and can therefore be used as backup in the unlikely event that any one of the three primary RWAs fails. The three primary RWA spin axes are oriented 120° apart when they are projected on the spacecraft x-y plane. All three spin axes are 54.73° (or $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$) off the spacecraft's +Z-axis. Figure 2 depicts both the location and orientation of the four RWAs.

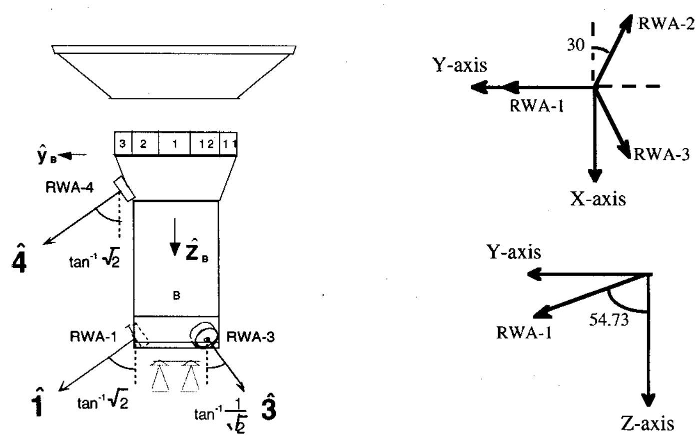

Figure 2 RWA Locations and Orientation

Knowing the orientations of the three RWAs relative to the XYZ body frame, we can determine a transformation matrix [T], from the RWA coordinate frame to the spacecraft XYZ body coordinate frame.

$$[T] = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Note that for the Cassini spacecraft, [T] is an orthogonal matrix. The RWAs are required to be mounted parallel to their respective specified orientation to within 0.5° (3σ per axis)[1]. This inaccuracy is considered negligible and the T matrix is assumed to be known "exactly" in this study.

Define a new vector, $\vec{\omega}_{RWA}$, which is a stack of the angular rates of the three RWAs. It has two components: the rate of the spacecraft and the spin rate of the RWAs with respect to the spacecraft. It is easiest to add these rates in each RWA coordinate frame and then to translate the total angular momentum of the RWAs from the RWA frame to the spacecraft XYZ body coordinate frame, and then finally to the inertial J_{2000} coordinate frame: $\vec{\omega}_{RWA}$(in RWA coordinates) = $[T]^{-1}\vec{\Omega}+\vec{\rho}$. Then to find $\vec{H}_{RWA}$ in body coordinates, we simply need to multiply $\vec{\omega}_{RWA}$ by the inertia matrix for the RWAs, and then multiply by the transformation matrix [T]: $\vec{H}_{RWA}$(in body coordinates) $= [T][I_{RWA}][[T]^{-1}\vec{\Omega}+\vec{\rho}]$. In this equation, $[I_{RWA}]$ is a diagonal matrix whose components are the three almost identical moments of inertia of RWA_1, RWA_2, and RWA_3. Invoking the following approximation: $[T][I_{RWA}][T]^{-1}\vec{\Omega} \approx [I_{RWA}]\vec{\Omega}$, we have:

$$\vec{H}_{RWA} \text{ (in body coordinates)} \approx [I_{RWA}]\vec{\Omega} + [T][I_{RWA}]\vec{\rho} \tag{2}$$

The total angular momentum of the spacecraft in body coordinates simply sums together the component from the spacecraft rate and the component from the RWAs.

$$\vec{H}_{Total}\text{(in body coordinates)} = [I_{SC} + I_{RWA}]\vec{\Omega} + [T][I_{RWA}]\vec{\rho}$$

But I_{RWA} is about four orders of magnitude smaller than I_{SC}. As such, we have:

$$\vec{H}_{Total}\text{(in body coordinates)} = [I_{SC}]\vec{\Omega} + [T][I_{RWA}]\vec{\rho} \tag{3}$$

Multiplying Equation (3) by the inverse of the transformation matrix [P] gives us the total angular momentum vector in the inertial J_{2000} coordinate frame. This vector is approximately conserved over a spacecraft slew.

$$\vec{H}_{Total}\text{(in inertial coordinates)} = [P]^{-1}[I_{SC}]\vec{\Omega} + [P]^{-1}[T][I_{RWA}]\vec{\rho} \tag{4}$$

The spacecraft is quiescent just prior to the slew, with all angular rates (ω_x, ω_y, and ω_z) approximately zero. As such, the initial angular momentum vector is given by:

$$\vec{H}_{Total}(0) = [P(0)]^{-1}[T][I_{RWA}]\vec{\rho}(0) \tag{5}$$

Where Variable(0) implies the initial value of that Variable (time = 0). As described in Appendix A, the environmental torque acting on the spacecraft is very small. For all practical purposes, one can assume that the total angular momentum of the spacecraft is conserved over the course of a spacecraft slew. This implies that the initial angular momentum from Equation (5) is equal to the angular momentum throughout the time of the slew.

$$[P(t)]^{-1}[I_{SC}]\vec{\Omega}(t) + [P(t)]^{-1}[T][I_{RWA}]\vec{\rho}(t) \approx [P(0)]^{-1}[T][I_{RWA}]\vec{\rho}(0) \tag{6}$$

Now consider the special case in which we slew the spacecraft one axis at a time. In this case, the rate components about the other two axes go to zero. For example, for a slew about the X-axis, Equation (6) becomes:

$$[I_{SC}]\begin{bmatrix} \omega_x(t) \\ 0 \\ 0 \end{bmatrix} = [P(t)][P(0)]^{-1}[T][I_{RWA}]\vec{\rho}(0) - [T][I_{RWA}]\vec{\rho}(t) \tag{7}$$

Denote the right hand side of this equation by a new vector Q(t). Q(t) = $[Q_x, Q_y, Q_z]^T$. Using this notation, the first component of the vector-matrix Equation (7) is:

$$I_{xx} * \omega_x(t) = Q_x(t) \tag{8}$$

In Equation (8), both ω_x and Q_x are functions of time. In other words, there are many data points available at time steps throughout the slew. Therefore, if we define a vector of data points for each, we can use a least square approach to obtain the best estimate for I_{xx}:

$$[\omega_x] = [\omega_x(t=0), \omega_x(t=1), \omega_x(t=2), \ldots, \omega_x(t=N-1)]^T$$

$$[Q_x] = [Q_x(t=0), Q_x(t=1), Q_x(t=2), \ldots, Q_x(t=N-1)]^T$$

$$\hat{I}_{xx}(\text{least - square estimate}) = [\omega_x{}^T \omega_x]^{-1} \omega_x{}^T Q_x$$

This process can be repeated for I_{yx} and I_{zx} using Q_y and Q_z respectively. The entire process can then be repeated for slews about the y and z-axes as well. This process will give one estimate for each of the Moments of Inertia (MOI) and two estimates for each one of the Products of Inertia (POI) (I_{xz} and I_{zx} which should be identical since the inertia matrix is by definition symmetrical). The two POI estimates can be averaged together to obtain the best estimate.

An alternative to the approach described above is to estimate all six independent components of the inertia matrix simultaneously using all the slew data (X, Y, and Z-axis slews) at the same time. The "axis-by-axis" approach described above was used because of its relative simplicity.

Input Data

To date only one maneuver has been done with the Cassini spacecraft using the RWAs. This maneuver was done on the seventy-fifth day of this year (March 15, 2000) and lasted from 00-075/17:20:00.461 to 00-075/19:19:56.413. As seen in Figure 3, this maneuver consisted of a slew about the Y-axis, followed by a slew about the X-axis, another slew about the Y-axis, a slew about the Z-axis, and finally a very small slew about the Y-axis. Telemetry data for the following quantities are available over the entire slew duration, at a frequency of once every four seconds: q_1, q_2, q_3, q_4, ω_x, ω_y, ω_z, ω_1, ω_2, and ω_3. Figures 3 and 4 depict the spacecraft per-axis angular rates and the RWA spin rates for the time period used.

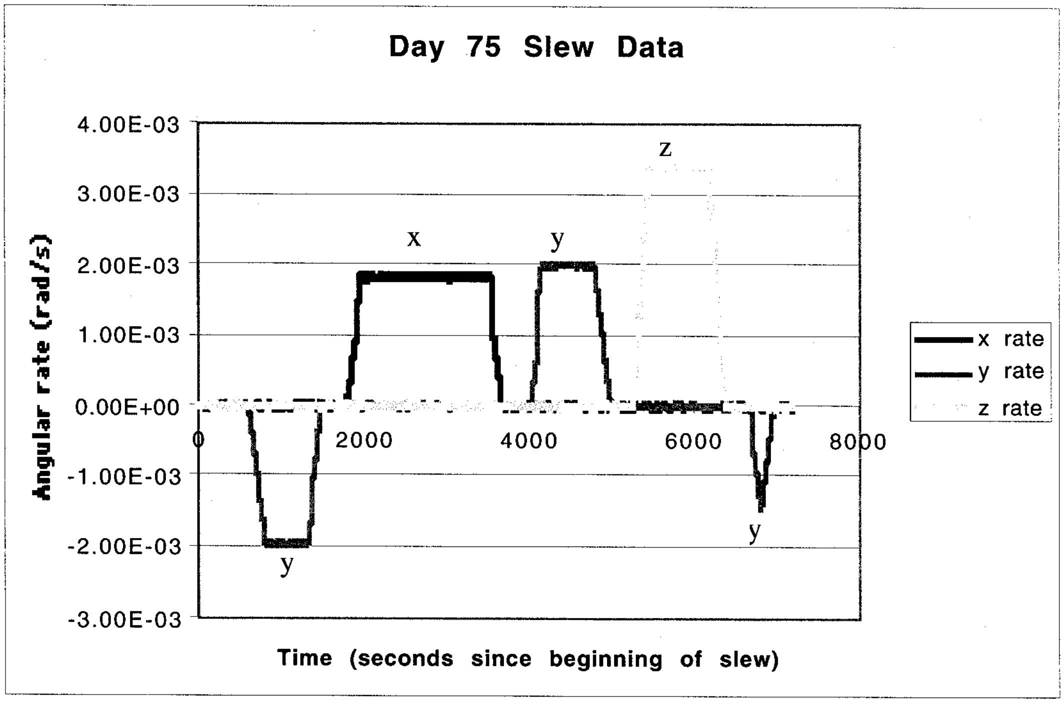

Figure 3 Spacecraft Per-axis Angular Rates

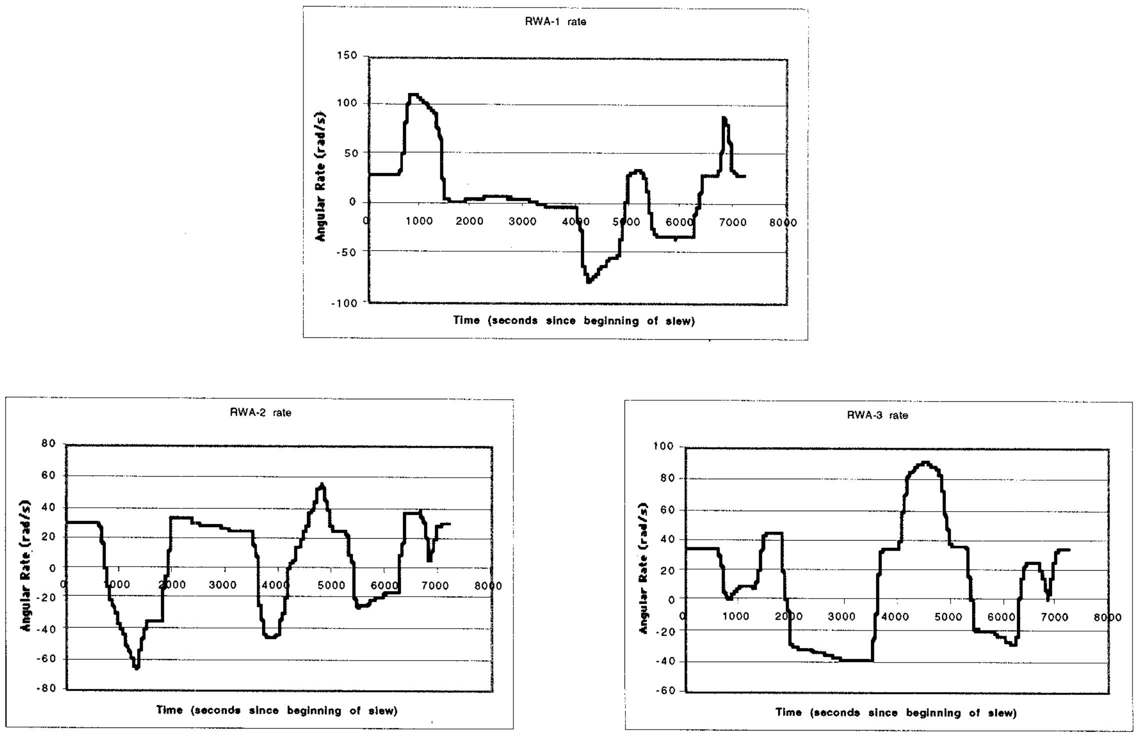

Figure 4 RWA Spin Rates

Results

The data from March 15, 2000 (DOY-75) was analyzed using the proposed methodology. The resulting best estimate for the inertia matrix of the spacecraft was:

$$I_{SC} = \begin{bmatrix} 8655.2 & -144 & 132.1 \\ -144 & 7922.7 & 192.1 \\ 132.1 & 192.1 & 4586.2 \end{bmatrix} kg \cdot m^2$$

When compared to the inertia matrix obtained using the existing method (see Equation (1)), our result validates the current method, as they are reasonably close. The MOI (I_{xx}, I_{yy}, and I_{zz}) resulting from the least squares method are consistently lower than their counterparts from the current method by at most 3%. This could point to a bias in the current method and/or in the estimate of the spacecraft inertia matrix prior to launch. This is entirely possible because the knowledge requirement for the MOI of the "dry" spacecraft is ±10%.[2] In addition, the POI (I_{xy}, I_{xz}, and I_{yz}) resulting from the least squares method described above are within 40 kg-m^2 of those resulting from the current method. The magnitude of the POI resulting from using the method described above are all larger than their counterparts from using the existing method, which again could be evidence of a bias. The knowledge requirement for the POI of the "dry" spacecraft is ±75 kg-m^2.[2]

The residuals are defined as:

$$R_{xx}(t) \equiv Q_x(t) - \hat{I}_{xx}\omega_x(t)$$
$$R_{yx}(t) \equiv Q_y(t) - \hat{I}_{yx}\omega_x(t)$$
$$R_{zx}(t) \equiv Q_z(t) - \hat{I}_{zx}\omega_x(t)$$

These residuals were plotted for the inertia matrix estimates obtained using the DOY-75 data and the method described above. These residuals can be seen in Figures 5 and 6. Note that very few of the residuals fall outside ±0.5 Nms. The knowledge requirement of the estimated angular momentum of the reaction wheels alone (i.e. $I_{RWA_{ii}} * \omega_i$) is set at 0.5 Nms. The RWA angular momentum however, was assumed to be a known constant in the derivation of this method, and therefore, the results are expected to be no better than that knowledge requirement.

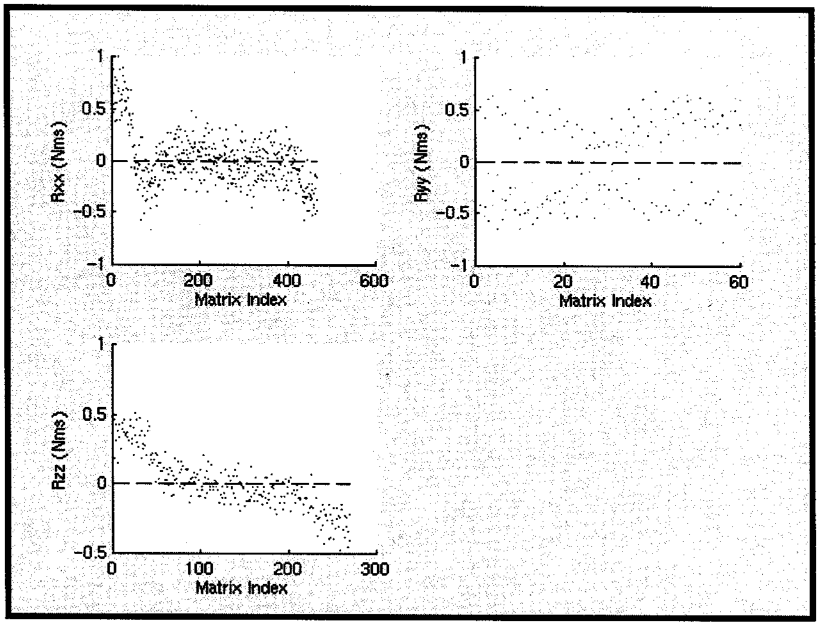

Figure 5 MOI Residuals

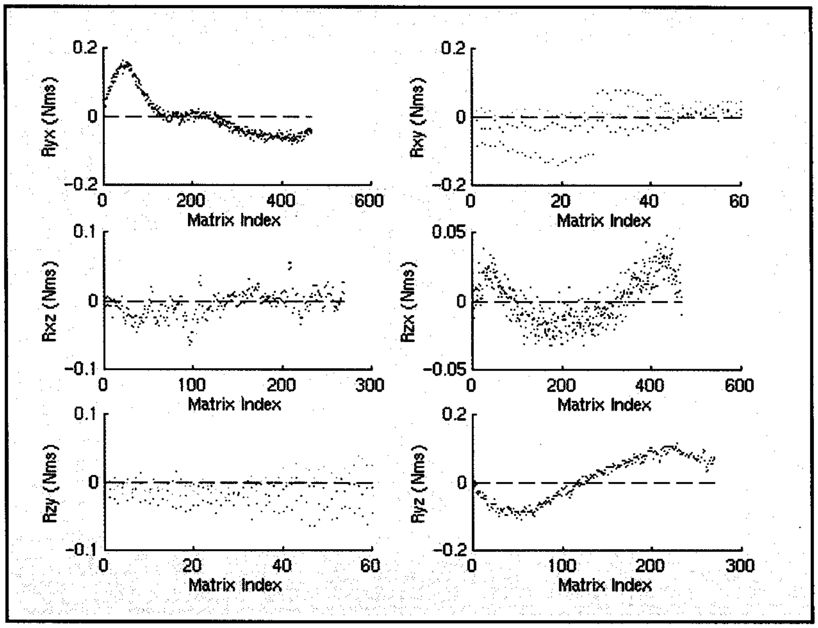

Figure 6 POI Residuals

Conclusion

The least square estimate obtained through the conservation of angular momentum method described above, from DOY-75 data, agrees closely with that determined by the existing method. This validates the current method and estimate, and provides a new method to estimate the spacecraft inertia matrix inflight. Using this method, the moments and products of inertia could be easily estimated whenever telemetry data associated with slewing the spacecraft by the RWAs is available. This methodology has been proven by its successful application on the Cassini spacecraft. The same methodology could also be used on other spacecraft with reaction wheel control.

References

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Appendix A

Second-order Effects that impact the Validity of the Methodology

Several second-order effects that might impact the validity of the proposed methodology and the accuracy of the results obtained have been investigated. The first effect investigated was the impact of the assumption that the external environmental torque acting on the spacecraft was negligibly small. During the Cruise phase of the Cassini mission, when the spacecraft is far from any planets (e.g. Jupiter or Saturn), there are only two main sources of external torque acting on the spacecraft. The first is the solar radiation torque and the second is the radiation torque from the power generator.

The total external torque acting on the spacecraft on a given day can be estimated using telemetry data from the spacecraft. For DOY-75 (the day the spacecraft was slewed by the three RWAs), the largest per-axis external torque was that about the spacecraft's X-axis, and was less than $1.5\mathrm{x}10^{-5}$ Nm. The five spacecraft per-axis slews depicted in Fig. 3 lasted for a total of approximately 7000 seconds, although each individual slew was performed over only about 1000-3000 seconds. Therefore, the order of magnitude of the change in angular momentum due to both the solar radiation torque and that from the power generator is about $1.5\mathrm{x}10^{-5} \times 3000 \approx 0.045$ Nms. This value is very small when compared with those due to either the spacecraft angular rates or the RWAs' spin rates. Hence, the assumption of "zero" external torque acting on the spacecraft is validated, and the total spacecraft angular momentum vector is conserved over the course of the spacecraft slew.

Another second-order effect analyzed is the impact of the filtered RWA rates (ω_1, ω_2, ω_3) used in the analysis. The RWA rates are estimated onboard by passing the raw tachometer data through a first-order filter. The resultant filtered spin rates (ω_1, ω_2, ω_3) are used in estimating the inertia tensor. This filtering smoothed out the raw tachometer data but it leads to small errors in the estimated spin rates. The impact of this filtering was examined by attempting to "reconstruct" the true spin rates using the filtered rates. When the spacecraft inertia tensor results estimated using the "with reconstruction" and "without reconstruction" approach were examined, the one made with the reconstructed RWA rates did not lead to appreciable improvement. The reconstruction process does add a layer of uncertainty to the analysis however, and it was therefore decided to use the simplest "without reconstruction" approach.

The effect of knowledge errors in the values assumed to be known constants was also studied. These knowledge errors vary in degree among the "known" values, however no value assumed to be known could ever be known completely. For example, the inertia matrix of the RWAs was measured during the qualification tests of the RWAs, and while it is known to a very accurate degree, there will always be some uncertainty associated with it. Other knowledge errors are more significant. For instance, the true orientation of the RWAs, which defines the transformation matrix [T], is accurate to with $\pm 0.5°$ (3σ per axis) of their nominal orientation. The per-axis spacecraft rates received from the spacecraft telemetry also have uncertainties in them (see Appendix B). Their 1σ estimation uncertainties vary from $1\text{x}10^{-5}$ to $2\text{x}10^{-5}$ rad/s. The RWA spin rates have a 1σ uncertainty of approximately 0.42 rad/s (see Appendix B). By propagating these uncertainties through the equations, we can estimate their contributions to the estimation uncertainty of the spacecraft inertia matrix. A preliminary analysis of the total estimation uncertainty of the spacecraft inertia tensor is given in Appendix B, and the 3σ estimation uncertainty of the spacecraft inertia tensor is:

$$3\sigma_{\hat{I}_{sc}} = \begin{bmatrix} 15.9 & 17.7 & 5.7 \\ 7.7 & 34.6 & 5.7 \\ 7.7 & 17.8 & 6.0 \end{bmatrix} kg \cdot m^2$$

The asymmetry of this estimation uncertainty matrix is explained in Appendix B.

Appendix B

Estimation Uncertainty of the Spacecraft Inertia Tensor

The inertia matrix of the spacecraft is estimated using a least-square approach. The approach uses telemetry data that themselves are "uncertain." Two major sources of uncertainties are that associated with the estimated spacecraft per-axis rates (σ_{ω_x}, σ_{ω_y}, σ_{ω_z}) and that associated with the estimated RWA rates (σ_ρ). The estimation uncertainty of the spacecraft inertia tensor due to these two sources is estimated in this Appendix. The following analysis does not take into account the estimation uncertainties associated with the spacecraft quaternion, the transformation matrix [T], and the RWA's inertia matrix [I_{RWA}].

The following approach illustrates the method in which the uncertainties listed above were carried through the equations using I_{xx} as an example. The estimation uncertainties for the other MOI and POI components could be found using a similar approach.

I_{xx} is least square estimated by solving N equations:

$$I_{xx}\omega_{x_j} = Q_{x_j} \quad \text{for } j = 1,2,\ldots,N$$

$$\hat{I}_{xx} = \frac{U}{V} = \frac{\sum_{j=1}^{N}\omega_{x_j}Q_{x_j}}{\sum_{j=1}^{N}\omega_{x_j}^2}$$

Where $\hat{I}_{xx}$ is the least squares estimate of I_{xx}. For simplicity, let us denote the numerator and denominator of $\hat{I}_{xx}$ as U and V where: $U \equiv \sum_{j=1}^{N}\omega_{x_j}Q_{x_j}$ and $V \equiv \sum_{j=1}^{N}\omega_{x_j}^2$. From the last equation, we have:

$$d\hat{I}_{xx} = \left(\frac{1}{V}\right)dU + \left(-\frac{U}{V^2}\right)dV \tag{B1}$$

But dU and dV are given by the following expressions:

$$dU = \sum_{j=1}^{N}\left\{Q_{x_j}d\omega_{x_j} - \omega_{x_j}G_{x_j}d\rho_j + \omega_{x_j}B_{x_j}d\rho_0\right\} \text{ and } \quad dV = 2\sum_{j=1}^{N}\left\{\omega_{x_j}d\omega_{x_j}\right\} \tag{B2}$$

The G_x and B_x matrices given in these expressions are defined as follow: Let $[B(t)] \equiv [P(t)][P(0)]^{-1}[T][I_{RWA}]=[B_x^T\ B_y^T\ B_z^T]^T$, and $[G]=[T][I_{RWA}]=[G_x^T\ G_y^T\ G_z^T]^T$. Note that, G_x, and B_x, etc. are all 1-by-3 matrices. Accordingly, we have, $\vec{Q} = B\vec{\rho}_0 - G\vec{\rho}$. That is:

$$Q_x = B_x^T\vec{\rho}_0 - G_x^T\vec{\rho},$$

$$Q_y = B_y^T\vec{\rho}_0 - G_y^T\vec{\rho},$$

$$Q_z = B_z^T\vec{\rho}_0 - G_z^T\vec{\rho}.$$

Substitutions of (B2) into (B1) give:

$$d\hat{I}_{xx} = -\left(\frac{1}{V}\right)\sum_{j=1}^{N}Q_{x_j}d\omega_{x_j} + \left(\frac{1}{V}\right)\sum_{j=1}^{N}\left\{-\omega_{x_j}G_{x_j}d\rho_j + \omega_{x_j}B_{x_j}d\rho_0\right\} \tag{B3}$$

Using (B3), the estimation uncertainty of I_{xx} is given by:

$$P = \sigma_\omega^2\sum_{j=1}^{N}Q_{x_j}^2 + \sigma_\rho^2\sum_{j=1}^{N}\left\{\omega_{x_j}^2G_{x_j}G_{x_j}^T\right\} + \sigma_{\rho_0}^2\left\{\sum_{j=1}^{N}\omega_{x_j}B_{x_j}\right\}^2 \tag{B4}$$

and finally, $\sigma_{I_{xx}}^2 = P/\{\sum_{j=1}^{N}\omega_{x_j}^2\}^2$.

This entire process can be repeated for all other MOI and POI components. The value estimated is then tripled to report a 3σ value. Equation (B4) indicated that the "size" of the estimation uncertainty of the spacecraft inertia matrix could be reduced in the following two ways. Firstly, we could lengthen the time duration of those per-axis slews. Secondly, we could increase the peak (coast) rates of those per-axis slews (however, there are constraints on these rates due to other considerations).

The approximate magnitudes of the measurement uncertainties of the spacecraft per-axis rates are: $\sigma_{\omega x} \approx 2\times10^{-5}$ rad/s, $\sigma_{\omega y} \approx 2\times10^{-5}$ rad/s, and $\sigma_{\omega z} \approx 1\times10^{-5}$ rad/s. The size of the measurement uncertainty of the RWA spin rates could be derived using the knowledge requirement for the RWA angular momentum.[1] The angular momentum of each RWA is to be estimated by passing the raw RWA tachometer rate data through a 0.1-Hz first-order filter. The 3σ uncertainty requirement of the RWA angular momentum is 0.6 Nms.[2] In the actual design of the RWA controller,[3] we did better than the requirement. The RWA angular momentum is estimated with accuracy that is better than 0.2 Nms (3σ). Hence, $\sigma_{\rho} = \sigma_{\rho 0} = (0.2/0.16)/3 \approx 0.42$ rad/s, where 0.16 kg-m^2 is the nominal magnitude of the RWA's inertia. Using these estimated values for the component uncertainties, the 3σ uncertainty of the estimated inertia tensor is:

$$3\sigma_{\hat{I}_{sc}} = \begin{bmatrix} 15.9 & 17.7 & 5.7 \\ 7.7 & 34.6 & 5.7 \\ 7.7 & 17.8 & 6.0 \end{bmatrix} kg \cdot m^2$$

Note that unlike the inertia matrix itself, this uncertainty matrix is not symmetrical. There are two reasons for this asymmetry, both pertaining to the fact that the uncertainties associated with the POI are calculated using the same equations, but with different inputs. For example, in order to make the matrix symmetric, $\sigma_{I_{xy}}$ would have to equal $\sigma_{I_{yx}}$. The uncertainty associated with $\hat{I}_{xy}$ is calculated using the Y-axis slews while the uncertainty associated with $\hat{I}_{yx}$ is calculated using the X-axis slew. The duration of the X-axis slew and the Y-axis slew are different, leading to different uncertainty values. In addition, in (B3), σ_{ω_x} is used with the X-axis slew, while σ_{ω_y} and σ_{ω_z} are used with the Y-axis and Z-axis slew, respectively. Since these values are not necessarily the same (i.e. $\sigma_{\omega_x} \neq \sigma_{\omega_z}$), the result of (B3), and therefore the estimation uncertainties, are not necessarily the same.